The Collatz conjecture

This afternoon, lecture on termination in my course *Logic and Computation* (we're using ACL2 as a logic and theorem proving engine, and ACL2 requires a termination proof to admit a definition as a new axiom). And this allowed me to use the Collatz conjecture as an example of the non-triviality of establishing termination even for simple recursive functions.

The Collatz conjecture, also known as the 3n + 1 conjecture, the Ulam conjecture, the Syracuse problem, the hailstone sequence, is named after the German mathematician Lothar Collatz, who proposed it in 1937. The conjecture is that for every positive natural number a[0], the sequence a[0], a[1], a[2], a[3],... given by:

```
a[n+1] = 1           if a[n] is 1
a[n+1] = a[n] / 2    if a[n] is even
a[n+1] = 3a[n] + 1   if a[n] is odd
```

always eventually hits 1.

Casting this as a termination problem, of course, is just asking whether the function f given by

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f(1) = 1
f(n) = f(n / 2)      if n is even
f(n) = f(3n + 1)     if n is odd
```

terminates on all positive natural number inputs.

No one knows whether this is true or not. Experimental evidence indicates that yes, but that's almost meaningless. In particular, we know that conjecture holds for all inputs up to more than 10^16. A good overview can be found in Weisstein's MathWorld:

- [Collatz Problem](#)

Because it is so simple to state, and is extremely easy to understand, the problem has been the focus of attention for many amateur mathematicians and number-dabblers. In fact, I believe it to be almost a rite of passage to spend some time playing with the conjecture when first going through a math degree. (Thankfully, my own flirtation with the problem amounted to wasting no more than two weeks trying to formalize the "pattern" I saw in the sequences produced. Ah, the folly of youth...)

Naturally, professional mathematicians have also looked at the problem, and have come up with a surprising number of connections with seemingly unrelated topics. Best known is Jeffrey Lagarias, now at U. of Michigan, but before that researcher at AT&T Bell Labs.

- [Jeffrey C. Lagarias: 3x+1 problem and related problems](#)

Quite readable is his paper giving a survey of the problem from a mathematical standpoint:

- [The 3x+1 problem and its generalizations](#), by J. Lagarias.

Also of interest are his annotated bibliographies on the problem:

- [The 3x+1 problem: An annotated bibliography (1963--2000)](#), by J. Lagarias
- [The 3x+1 Problem: An Annotated Bibliography, II (2001-)](#), by J. Lagarias

Both can be found on the arXiv.

Now, I seem to remember someone establishing a connection between this conjecture and the distribution of prime numbers, but I have not been able to retrace this. If anyone knows what I am thinking of, please drop me a line.